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# Hopf algebra primitives in perturbation quantum field theory

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**Abstract**

The analysis of the combinatorics resulting from the perturbative expansion of the transition amplitude in quantum field theories, and the relation of this expansion to the Hausdorff series leads naturally to consider an infinite dimensional Lie subalgebra and the corresponding enveloping Hopf algebra, to which the elements of this series are associated. We show that in the context of these structures the power sum symmetric functionals of the perturbative expansion are Hopf primitives and that they are given by linear combinations of Hall polynomials, or diagrammatically by Hall trees. We show that each Hall tree corresponds to sums of Feynman diagrams each with the same number of vertices, external legs and loops. In addition, since the Lie subalgebra admits a derivation endomorphism, we also show that with respect to it these primitives are cyclic vectors generated by the free propagator, and thus provide a recursion relation by means of which the $(n + 1)$-vertex connected Green functions can be derived systematically from the $n$-vertex ones.


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## 1. Introduction

It has been conjectured [17] that because of the interplay of Quantum Mechanics and General Relativity at the Planck scale, space–time ought to be regarded as a derived concept

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whose structure should follow from the properties of Quantum Field Theory. Perturbative quantum field theory (PQFT), albeit its conceivable limitations at distances of the order of the Planck length, is the only computational tool available at present, and so the elucidation of the mathematical structures behind that theory is a suggestive first step in pursuing this line of thought.

One of these structures, which is based on the original work of Kreimer [1] and further developed in [2–5], is now generally known as the Hopf Algebra of Renormalization and it provides the underlying mathematics behind the Forest Formula in the process of renormalization. Basically this Hopf algebra can be represented by Feynman diagrams or decorated rooted trees, where decorations are one-particle irreducible (1PI) divergent diagrams without subdivergences. Other Hopf algebras related to rooted trees and to the Hopf Algebra of Renormalization have been discussed in the literature, such as the vector space Hopf algebra of rooted trees of Grossman and Larson [6]. The connection of this algebra with the algebra of Kreimer and Connes was analyzed in [7] and more recently revised in [8]. For a formulation in terms of a single mathematical construction of the several Hopf algebras described by rooted trees see van der Laan [9].

The essential point of the Kreimer–Connes formalism is that *given that a theory is renormalizable* an appropriately defined twisted antipode, based on the minimal subtraction scheme of renormalization, generates the counterterms corresponding to the BPHZ Forest Formula and the antipode axiom provides a systematic procedure for deriving the physically correct and finite expression for a given diagram.

However, the mere fact that the twisted antipode axiom, or for that matter the Forest Formula, provide a finite answer does not suffice to make the theory physical. It is crucial, in order that the theory be renormalizable, that the resulting counterterms are of the same form as those in the original Lagrangian and that they can be absorbed into the bare parameters of the renormalized Lagrangian in a consistent manner. Generally this is not possible, and in such a case the theory is described as nonrenormalizable.

Of course if we know a priori that the theory is renormalizable then the Hopf algebra of decorated rooted trees or of Feynman diagrams remains most valuable both as the mathematical structure behind the Forest Formula as well as for a systematic construction of renormalized Green functions.

Another Hopf algebra related to rooted trees by a canonical mapping is the algebra of normal coordinates that was recently discussed in [10]. In that work undecorated rooted trees were considered, and the relevance of normal coordinates to the concept of $k$-primitiveness as well as their role in the process of renormalization was analyzed. It was also shown there that for undecorated ladder trees, or to that effect for nonbranched trees with only one decoration (such as in rainbow diagrams), the renormalization of the associated normal coordinates is a one step procedure. However, when the diagrams for a theory involve more than one decoration (as is usually the case) ladder normal coordinates are in general no longer primitive, even though one can still expect them to posses a milder pole structure than that present in the rooted tree coordinates.

Some of the pertinent questions left open in the above cited paper concerned the physical interpretation of the normal coordinates and whether perturbation theory could be formulated directly in terms of them without having to go first through the algebra of rooted trees or Feynman diagrams. These questions provided some partial motivation for the present work where we investigate some additional Hopf algebra structures that can be associated naturally to the Hausdorff series expansion of PQFT. Specifically, we show here that the power sum symmetric functionals of these perturbative expansions are elements of a free Lie subalgebra as well as Hopf primitives of its enveloping free algebra. Moreover these primitives can be expressed as linear combinations of Hall polynomials and, diagrammatically, as Hall trees. We show also that a Hall tree corresponds to sums of Feynman diagrams, each with the same number of vertices, external legs and loops.

In addition, since the Lie subalgebra admits a derivation endomorphism, we further show that—with respect to it—these primitives are cyclic vectors generated by the free propagator, and thus provide a recursion relation by means of which the $(n+1)$-vertex connected Green functions can be derived systematically from the $n$-vertex ones.

Lastly we show that the Hopf primitives considered here are normal coordinates resulting from a canonical mapping applied to the Poincaré–Birkhoff–Witt basis constructed from the complete homogeneous symmetric functionals, which appear also naturally in the Hausdorff series expansion of PQFT. This combinatorics is reminiscent of the one appearing in [10], but the possibility of a deeper relationship between the two Hopf algebras, and therefore a possible physical interpretation for the normal coordinates in [10], requires further investigation.

The paper is structured as follows: In Section 2 we begin with a brief review of the main steps that lead to the perturbative expansion of the transition amplitude, and describe its relation to the Hausdorff series and to the associated Hopf algebras for which the power sum functionals are primitives. In Sections 2.1.1–2.1.3 the free algebra related to these structures is discussed and it is shown that the functional primitives are linear combinations of the Hall polynomials that generate the Lie subalgebra of our free algebra. We also show that this Lie subalgebra admits a derivation endomorphism with respect to which all the primitives are cyclic vectors generated by the free propagator. The diagrammatic representation of the primitives in terms of Hall trees is also discussed in this section and shown to give a clear image of this cyclicity and of the iteration process by means of which the $(n+1)$-vertex connected Green functions can be constructed from the $n$-vertex ones. Section 3 is devoted to a discussion of our main results and possible lines of future related research. We also give there an explicit and heuristic argument, based on a Birkhoff decomposition of the Hopf algebra considered here, which we believe helps to stress some of the points made in this introduction.

## 2. Algebraic structures in perturbation quantum field theory

Let us begin with a brief summary of the essential steps in PQFT leading to the Green functions, with the dual purpose of making our presentation self-contained as well as for identifying the basic mathematical and physical entities to which the Hopf algebras considered here are related.

For an arbitrary field theory the Euclidean transition amplitude (the formulation in Minkowski space is achieved by analytic continuation) is given by

$$W_{\mathrm{E}}[\mathbf{J}] = N \int \mathcal{D}\mathbf{\Phi}\, \mathrm{e}^{-\int \mathrm{d}^d x[\mathcal{L}_0+\mathcal{L}_{\mathrm{int}}-\mathbf{J}\cdot\mathbf{\Phi}]}. \tag{1}$$

Here $\mathbf{\Phi}$ denotes the set of fields appearing in the theory, and $\mathbf{J}$ denotes the set of arbitrary currents introduced to drive each field.[1] Using functional derivatives the amplitude (1) can be rewritten as

$$W_{\mathrm{E}}[\mathbf{J}] = \mathrm{e}^{-\langle \mathcal{L}_{\mathrm{int}}(\delta/\delta \mathbf{J}_x)\rangle_x}\, \mathrm{e}^{-Z^0[\mathbf{J}]} W_0[0], \tag{2}$$

where $\mathcal{L}_{\mathrm{int}}(\delta/\delta\mathbf{J}_x)$ is the Lagrangian of interaction written in terms of functional derivatives of the field currents $\mathbf{J}(x)$, and $W_0[\mathbf{J}] = \mathrm{e}^{-Z^0[\mathbf{J}]}W_0[0]$ is the free generating functional. The symbol $\langle\,\rangle_x$ stands for integration over the variable $x$ (after acting to the right with the functional derivative). Note that the functional derivatives with respect to the currents act according to the Leibniz rule on the term $\mathrm{e}^{-Z^0[\mathbf{J}]}$, and functional derivatives that go through to the right of that term cancel when acting on $W_0[0]$. Thus here $W_{\mathrm{E}}[\mathbf{J}]$ is a functional and not an operator.

To simplify our exposition we shall consider the neutral scalar $\varphi^4$ theory in Euclidean four dimensions when doing explicit calculations. It is well known that this theory is renormalizable and a clear discussion of the steps leading to its renormalization may be found in [11]. For this case the transition amplitude in (1) reduces to

$$W_{\mathrm{E}}[J] = N \int \mathcal{D}\varphi\, \mathrm{e}^{-\int \mathrm{d}^4x[(1/2)\partial_\mu\varphi\partial_\mu\varphi + (1/2)m^2\varphi^2 + V(\varphi) - J\varphi]}, \tag{3}$$

and (2) becomes

$$W_{\mathrm{E}}[J] = \mathrm{e}^{-\langle V(\delta/\delta J_x)\rangle_x}\, \mathrm{e}^{-Z^0[J]} W_0[0], \tag{4}$$

where

$$\left\langle V\left(\frac{\delta}{\delta J_x}\right)\right\rangle_x = \int \mathrm{d}^4x \frac{\lambda}{4!}\frac{\delta^4}{\delta J_x^4}, \tag{5}$$

$$W_0[0] = N \int \mathcal{D}\varphi\, \mathrm{e}^{-\int \mathrm{d}^4x[(1/2)\partial_\mu\varphi\partial_\mu\varphi + (1/2)m^2\varphi^2]}, \tag{6}$$

$$Z^0[J] = \tfrac{1}{2}\langle J(x)\Delta_{xy}J(y)\rangle_{xy} \tag{7}$$

and

$$\Delta_{xy} = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p \frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{p^2+m^2} \tag{8}$$

is the Feynman propagator in four-dimensional Euclidean space.

Writing $W_{\mathrm{E}}[J] = \mathrm{e}^{-Z_{\mathrm{E}}[J]}$ and rearranging (4) results in

$$Z_{\mathrm{E}}[J] = -\ln\, W_0[0] + Z^0[J] - \ln(1 + \mathrm{e}^{Z^0[J]}(\mathrm{e}^{-\langle V(\delta/\delta J)\rangle} - 1)\, \mathrm{e}^{-Z^0[J]}(1)), \tag{9}$$

[1] Following Ref. [11] we use the notation $W_{\mathrm{E}}[J]$ for the generating functional of connected and disconnected graphs, and $Z_{\mathrm{E}}[J]$ for the generating functional of connected graphs, sometimes the opposite notation is used in other works.

where, in order to make both sides of the above equation consistent, we have explicitly included the action on the identity function of the operator inside the logarithm, so as to cancel derivations to the right of $e^{-Z^0[J]}$. Now let

$$\sigma(\lambda) := e^{Z^0[J]}(e^{-\langle V(\delta/\delta J)\rangle} - 1)\, e^{-Z^0[J]}(1) = \sum_{k\geq 1} \lambda^k S_k[J], \tag{10}$$

where the second equality is the formal power series expansion of the first one in terms of the coupling constant $\lambda$ in the potential. If we further define

$$\psi(\lambda) := \sum_{k\geq 1} \lambda^k \psi_k[J] = \ln\left(1 + \sum_{k\geq 1} \lambda^k S_k[J]\right), \tag{11}$$

it then readily follows from (9) that

$$Z_{\mathrm{E}}[J] = -\ln W_0[0] + Z^0[J] - \sum_{k\geq 1} \lambda^k \psi_k[J]. \tag{12}$$

Note that in the theory of symmetric functions [12,13] an expression like (11) relates the complete homogeneous symmetric functions to the so called power sum symmetric functions or Schur polynomials, so we can use the combinatorics of that theory to write down the explicit (invertible) relation between the functionals $\psi_k[J]$ and the $S_i[J]$, as homogeneous polynomials of order $k$ in the latter. This is given by

$$\psi_k[J] = \sum_{|I|=k} (-1)^{l(I)-1} \frac{S^I}{l(I)}, \tag{13}$$

where $I$ is a composition $I = (i_1, \ldots, i_r)$ of nonnegative integers, $|I| = \sum_k i_k$ is its weight, $l(I) = r$ is its length and $S^I = S_{i_1} S_{i_2} \ldots S_{i_r}$. In terms of the Feynman diagrammatic representation, the $\psi_k[J]$ correspond to linear combinations of connected graphs each with $k$ vertices.

Defining the operator

$$\begin{aligned} X &= -\frac{1}{\lambda} e^{Z^0[J]} \left(\left\langle V\left(\frac{\delta}{\delta J_{x_1}}\right)\right\rangle_{x_1}\right) e^{-Z^0[J]} \\ &= -\frac{1}{\lambda} \sum_{n\geq 0} \frac{1}{n!} \mathrm{ad}(Z^0[J])^n \left(\left\langle V\left(\frac{\delta}{\delta J_{x_1}}\right)\right\rangle_{x_1}\right), \end{aligned} \tag{14}$$

where the adjoint operator in the second equality is defined as the right normed bracketing: $\mathrm{ad}(b)^n(a) \equiv [b, [b, \ldots, [b, a]], \ldots]$, one can verify that the functionals $S_k$ introduced above are given by the recursion relations

$$S_1[J] = X(1), \qquad S_n[J] = \frac{1}{n} X(S_{n-1}[J])(1), \tag{15}$$

where $X(1)$ denotes the action of the operator (14) on the identity function.

In the same way that $X$ generates a recursion relation for the functionals $S_k$, we will show below that a derivation operator can be defined which when acting on the $\psi_k[J]$ leads to expressions analogous to (15). We first analyze some of the algebraic structures behind the operators occurring in (4) and (9).

## 2.1. Algebraic structures

As mentioned in Section 1, one of our goals in this paper is to investigate the relevance to the process of renormalization of Hopf algebra primitives that occur naturally at a prediagram stage in PQFT. Therefore, these primitives must be related to the two entities appearing in the perturbation expansion of the transition amplitude, i.e. the complete homogeneous symmetric functionals and the power sum symmetric functionals.

From the theory of symmetric functions we know that one can associate respective Hopf algebra structures to these two sets of functionals for which the $\psi_k[J]$ are primitives. In addition to the above, there is a free algebra generated by the operator $\langle V\rangle$ and the functional $Z^0[J]$ for which the $\psi_k[J]$ are also primitives. We describe these three Hopf algebras which are related by the following diagram

$$
\begin{tikzcd}
K<\{S_k\}> \simeq K<\{\psi_k\}> \subset & K<\{\Psi_k\}> \arrow[r, hook] & K<Y> \\
& \mathfrak{L}_K(\{\Psi_k\}) \arrow[u, phantom, "\bigcup"] \arrow[r, hook] & \mathfrak{L}_K(Y) \arrow[u, phantom, "\bigcup"] \\
& \psi_k \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

### 2.1.1. The free algebra $K\langle Y\rangle$ and its Hopf algebra structure

Let $K\langle Y\rangle$ be the unital free associative $K$-algebra over a field of characteristic zero (including $\mathbb{Q}$) and generated by the two-letter alphabet $Y = \{Z^0[J], \langle V(\delta/\delta J_x)\rangle_x\}$ of non-commuting variables, with concatenation as multiplication and unit (neutral) element $\mathbf{1}$ the empty word. Let $\mathfrak{L}_K(Y)$ be the infinite dimensional free Lie algebra on $Y$ where its elements are submodules of $K\langle Y\rangle$ with Lie bracket as multiplication. $K\langle Y\rangle$ is the enveloping algebra of $\mathfrak{L}_K(Y)$. We can give $K\langle Y\rangle$ a Hopf algebra structure [14] by defining a primitive coproduct on the alphabet letters:

$$\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}, \tag{16}$$

$$\Delta(Z^0[J]) = \mathbf{1}\otimes Z^0[J] + Z^0[J]\otimes\mathbf{1}, \tag{17}$$

$$\Delta(\langle V\rangle) = \mathbf{1}\otimes\langle V\rangle + \langle V\rangle\otimes\mathbf{1} \tag{18}$$

and extending it to words by the connection axiom. The antipode is given by

$$S(a) = -a \quad (a = \langle V\rangle \text{ or } Z^0[J]), \tag{19}$$

$$S(\mathbf{1}) = \mathbf{1} \tag{20}$$

and is extended to words by the anti-homomorphism

$$S(a_1\ldots a_n) = S(a_n)\ldots S(a_1). \tag{21}$$

The counit map $\varepsilon : K\langle Y\rangle \to K$ is defined on the generating letters by

$$\varepsilon(a) = 0 \quad (a = \langle V\rangle \text{ or } Z^0[J]), \tag{22}$$

$$\varepsilon(\mathbf{1}) = 1 \tag{23}$$

and is extended to words by the connection axiom. All the elements (Lie polynomials) $P \in \mathfrak{L}_K(Y)$ are primitives of this Hopf algebra:

$$\Delta(P) = \mathbf{1} \otimes P + P \otimes \mathbf{1} \tag{24}$$

and

$$S(P) = -P, \tag{25}$$

$$\varepsilon(P) = (P, \mathbf{1}), \tag{26}$$

where $(P, \mathbf{1})$ is the coefficient in $P$ of the unit element.

We introduce now on $K\langle Y\rangle$ a derivation which maps $\langle V(\delta/\delta J_x)\rangle_x$ onto 0, and $Z^0[J]$ onto $X$ by means of the operator

$$D = -X\frac{\partial}{\partial Z^0[J]}. \tag{27}$$

Clearly $D$ is an endomorphism on $K\langle Y\rangle$.

Since $Z^0[J]$ and $\langle V(\delta/\delta J_x)\rangle_x$ do not commute, the action of the derivation $D$ on an arbitrary function of $Z^0[J]$ is given by [14]:

$$D(f(Z^0[J])) = \sum_{k\geq 1}\frac{1}{k!}\mathrm{ad}(Z^0[J])^{k-1}(DZ^0[J])\, f^{(k)}(Z^0[J]) \tag{28}$$

and, in particular,

$$\begin{aligned} D(\mathrm{e}^{-Z^0[J]}) &= \sum_{k\geq 1}\frac{1}{k!}\mathrm{ad}(-Z^0[J])^{k-1}(X)\,\mathrm{e}^{-Z^0[J]} = \left(\frac{\mathrm{e}^{\mathrm{ad}(-Z^0[J])}-1}{\mathrm{ad}(-Z^0[J])}\right)(X)\,\mathrm{e}^{-Z^0[J]} \\ &= \mathrm{e}^{-Z^0[J]}(X)\,\mathrm{e}^{Z^0[J]}\cdot\mathrm{e}^{-Z^0[J]} = -\frac{1}{\lambda}\left\langle V\left(\frac{\delta}{\delta J_x}\right)\right\rangle_x \mathrm{e}^{-Z^0[J]}, \end{aligned} \tag{29}$$

where in going from the third to the last equality we have made use of (14). Moreover, since $D(\langle V(\delta/\delta J_x)\rangle_x) = 0$, we immediately get

$$\lambda^n D^n(\mathrm{e}^{-Z^0[J]}) = \left(-\left\langle V\left(\frac{\delta}{\delta J_x}\right)\right\rangle_x\right)^n \mathrm{e}^{-Z^0[J]}. \tag{30}$$

Using now once more the fact that $D$ is a derivation, we have that the exponential map $\mu = \mathrm{e}^{\lambda D}$ is a homomorphism of algebras and

$$\begin{aligned} \mu(\mathrm{e}^{-Z^0[J]}) &= \sum_{n\geq 0}\frac{\lambda^n}{n!}D^n(\mathrm{e}^{-Z^0[J]}) = \sum_{n\geq 0}\frac{1}{n!}\left(-\left\langle V\left(\frac{\delta}{\delta J_x}\right)\right\rangle_x\right)^n \mathrm{e}^{-Z^0[J]} \\ &= \mathrm{e}^{-\langle V(\delta/\delta J_x)\rangle_x}\,\mathrm{e}^{-Z^0[J]}. \end{aligned} \tag{31}$$

Furthermore, since $\mu$ is a continuous homomorphism,

$$\mathrm{e}^{-\langle V(\delta/\delta J_x)\rangle_x}\,\mathrm{e}^{-Z^0[J]} = \mu(\mathrm{e}^{-Z^0[J]}) = \mathrm{e}^{\mu(-Z^0[J])} = \exp\left(\sum_{n\geq 0}\frac{\lambda^n}{n!}D^n(Z^0[J])\right), \tag{32}$$

where

$$D^n(Z^0[J]) = (-1)^n\underbrace{\left(X\frac{\partial}{\partial Z^0[J]}\right)\cdots\left(X\frac{\partial}{\partial Z^0[J]}\right)}_{n}(Z^0[J]). \tag{33}$$

Writing

$$\mathrm{e}^{\boldsymbol{\Psi}} = \mathrm{e}^{-\langle V(\delta/\delta J_x)\rangle_x}\,\mathrm{e}^{-Z^0[J]}, \tag{34}$$

we obtain from (32) and (34) the Hausdorff series relation

$$\boldsymbol{\Psi} = Z^0[J] + \sum_{n\geq 1}\frac{\lambda^n}{k!}D^k(Z^0[J]). \tag{35}$$

Let us now define

$$\boldsymbol{\Psi}_k := \frac{1}{k!}D^k(Z^0[J]), \tag{36}$$

which exhibits the operators $\boldsymbol{\Psi}_k$ as cyclic vectors with respect to $D$ generated by $Z^0[J]$, and rewrite $X$, introduced in (14), as

$$X = -\frac{1}{\lambda}\sum_{j=0}^{d}\frac{1}{j!}\left[Z^0[J], \left\langle V\left(\frac{\delta}{\delta J_x}\right)\right\rangle_x\right]_j, \tag{37}$$

where $[Z^0[J], \langle V(\delta/\delta J_x)\rangle_x]_j$ is defined recursively by $[Z^0[J], \langle V(\delta/\delta J_x)\rangle_x]_j = [Z^0[J], [Z^0[J], \langle V(\delta/\delta J_x)\rangle_x]_{j-1}]$, $[Z^0[J], \langle V(\delta/\delta J_x)\rangle_x]_0 = \langle V(\delta/\delta J_x)\rangle_x$, and the upper index $d$ in the sum above is the degree of the functional derivative in $\langle V(\delta/\delta J_x)\rangle_x$ ($n = 4$ for the $\varphi^4$ theory).

It clearly follows from this and (33) that the cyclic vectors $\boldsymbol{\Psi}_k$ are elements of $\mathfrak{L}_K(Y)$ and, hence, primitive elements of the Hopf algebra $K\langle Y\rangle$.

In addition, since $D$ is also an endomorphism on $\mathfrak{L}_K(Y)$ there is a Hochschild cohomology associated with this algebra for which $D$ is a 1-cochain $D : \mathfrak{L}_K(Y) \to \mathfrak{L}_K(Y)$ with coboundary

$$bD(P) = (\mathrm{id}\otimes D)\Delta(P) - D(P)\otimes\mathbf{1}, \quad P\in\mathfrak{L}_K(Y). \tag{38}$$

Evidently $bD(P) = 0$, because of (24), so the Lie polynomials $P$, and in particular the $\boldsymbol{\Psi}_k$, are 1-cocycles for this cohomology.

Note that by applying both sides of (35) to the identity function and recalling (4) we recover (12), with the functional $\psi_k[J]$ given by

$$\psi_k[J] = -\frac{1}{k!}D^k(Z^0[J])(1). \tag{39}$$

Furthermore, since the $\psi_k[J]$ are made up by linear combinations of Lie monomials that do not cancel when acting from the left on the identity they are obviously also elements of $\mathfrak{L}_K(Y)$ and primitives of the Hopf algebra $K\langle Y\rangle$.

We can derive a recursion relation for the cyclic functionals $\psi_k[J]$ by observing that the right side of (39) can be written as

$$D^n(Z^0[J])(1) = (-1)^n \left( \underbrace{\left(X\frac{\partial}{\partial Z^0[J]}\right)\cdots\left(X\frac{\partial}{\partial Z^0[J]}\right)}_{n-1} X \right)(1). \tag{40}$$

In particular, for $n = 1$

$$-D(Z^0[J])(1) = \left(X\frac{\partial}{\partial Z^0[J]}(Z^0[J])\right)(1) = X(1) = \psi_1, \tag{41}$$

which is in agreement with (15) and (13).

To further see how to interpret the right side of (40) for $n \geq 2$, use (37) and note that in order to take into account the fact that the identity cancels functional derivatives acting on it, we must first apply the $n-1$ derivations in (40) to $X$ following the Leibnitz rule, evaluate each of the resulting derivations on $X$ inside the commutators by acting on the identity function according to (39), and finally act with the resulting bracket polynomial on the identity. Thus

$$\begin{aligned}\psi_2 &= -\frac{1}{2}\left(\left(X\frac{\partial}{\partial Z^0[J]}\right)X\right)(1)\\ &= \frac{1}{2\lambda}\left(X\frac{\partial}{\partial Z^0[J]}\right)\\ &\quad\times\left(\langle V\rangle - [\langle V\rangle, Z^0[J]] + \frac{1}{2}[[\langle V\rangle, Z^0[J]], Z^0[J]] + \sum_{j=3}^{d}\frac{1}{j!}[Z^0[J], \langle V\rangle]_j\right)(1)\\ &= \frac{1}{2\lambda}\left(-[\langle V\rangle, \psi_1] + \frac{1}{2}[[\langle V\rangle, \psi_1], Z^0[J]] + \frac{1}{2}[[\langle V\rangle, Z^0[J]], \psi_1] + \cdots\right)(1),\end{aligned} \tag{42}$$

$$\begin{aligned}\psi_3 &= \frac{1}{6}\left(\left(X\frac{\partial}{\partial Z^0[J]}\right)\left(X\frac{\partial}{\partial Z^0[J]}\right)X\right)(1)\\ &= -\frac{1}{6\lambda}\left(X\frac{\partial}{\partial Z^0[J]}\right)\\ &\quad\times\left(-[\langle V\rangle, X] + \frac{1}{2}[[\langle V\rangle, X], Z^0[J]] + \frac{1}{2}[[\langle V\rangle, Z^0[J]], X] + \cdots\right)(1)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{6\lambda}\left(-\left[\langle V\rangle, \left(\left(X\frac{\partial}{\partial Z^0[J]}\right)X\right)(1)\right]\right)\\
&\quad+\frac{1}{2}\left[\left[\langle V\rangle, \left(\left(X\frac{\partial}{\partial Z^0[J]}\right)X\right)(1)\right], Z^0[J]\right]\\
&\quad+\frac{1}{2}\left(\left[[\langle V\rangle, Z^0[J]], \left(\left(X\frac{\partial}{\partial Z^0[J]}\right)X\right)(1)\right]+[[\langle V\rangle,\psi_1],\psi_1]+\cdots\right)(1)\\
&= -\frac{1}{6\lambda}(2[\langle V\rangle,\psi_2]-[[\langle V\rangle,\psi_2],Z^0[J]]+[[\langle V\rangle,\psi_1],\psi_1]\\
&\quad-[[\langle V\rangle, Z^0[J]],\psi_2]+\cdots)(1). \qquad (43)
\end{aligned}
$$

Iterating on the above, we get the general recursion relation

$$\psi_{n+1}[J]=\frac{1}{n+1}D\psi_n[J]=-\frac{1}{n+1}\psi_1\frac{\partial}{\partial Z^0[J]}\psi_n. \qquad (44)$$

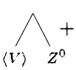
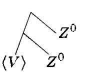
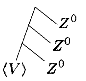
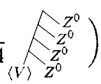
It should be recalled, however, that in the implementation of (44) one has to take $\psi_1 = -(1/\lambda)\sum_{j=1}^{d}(1/j!)[Z^0[J], \langle V(\delta/\delta J_x)\rangle_x]_j$, perform the derivations to the required order and then evaluate on the identity function.

We can express the above results in graphical form by making use of Hall trees. To this end recall [14] that each Hall tree $h$ of order at least 2 can be written as $h = (h', h'')$, where $h'$ and $h''$ are the immediate left and right subtrees, respectively, and such that the total ordering

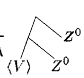
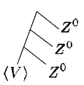
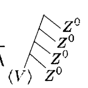
$$h < h'', \qquad h' < h'' \quad \text{and either } h' \in Y, \text{ or } h' = (x, y) \text{ and } y \geq h'' \qquad (45)$$

is satisfied. This ordering is lexicographical and is determined by the letters in the alphabet $Y$ that label the leaves. Also, each node in a tree corresponds to a Lie bracket and the foliage $f(h)$ of the tree is the canonical mapping defined by $f(a) = a$ if $a$ is in $Y$ and $f(h) = f(h')f(h'')$ if $h = (h', h'')$ is of degree $\geq 2$. Now, since a Hall word is the foliage of a unique Hall tree, and since for each Hall word $h$ there is a Lie polynomial $P_h$, it can be shown that these Hall polynomials form an infinite dimensional basis of the Lie algebra $\mathfrak{L}_K(Y)$ viewed as a $K$-module. Parenthetically, one also has that the decreasing products of Hall polynomials $P_{h_1}\dots P_{h_n}$, $h_1 \geq \cdots h_n$ form a basis of the free associative algebra $K\langle Y\rangle$.

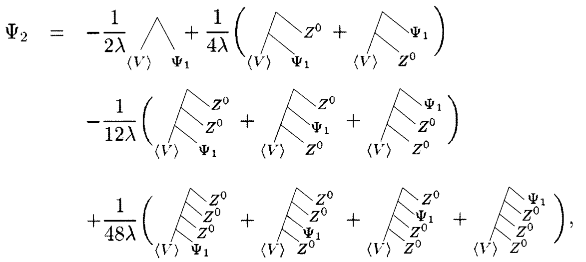
Consequently, using the ordering $\langle V\rangle < Z^0$ for our two letter alphabet $Y$ and the algorithm given in the proof of Theorem 4.9 in [14], we can always write the $\psi_i[J]$'s, as derived from (36) and (44), as linear combinations of Hall polynomials (equivalently Hall trees). Thus, for example

$\Psi_1 \equiv X = -\frac{1}{\lambda}\Big(\langle V\rangle -$  $+\frac{1}{2}$  $-\frac{1}{6}$  $+\frac{1}{24}$  $\Big),$ (46)

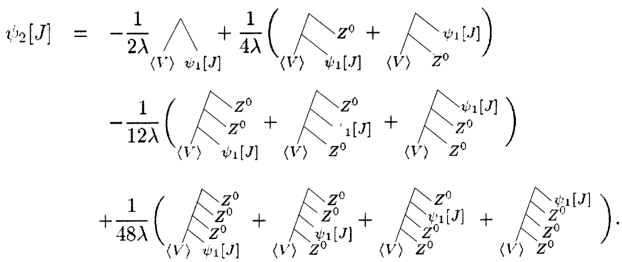
so

$\psi_1[J] = \Psi_1(1) = -\frac{1}{2\lambda}$  $+\frac{1}{6\lambda}$  $-\frac{1}{24\lambda}$  ; (47)

and



(48)



(49)

Note that (47) and (49) for the functionals $\psi_1[J]$ and $\psi_2[J]$ imply acting on the identity function from the left with the diagrams and replacing in addition the $\Psi_1$ in the foliage of (48) by $\psi_1[J]$. The corresponding Hall tree for $\psi_2[J]$ is obtained by grafting (47) onto each of the branches labeled with $\psi_1[J]$ and implementing the above mentioned algorithm. The procedure is then iterated to whatever order of the $\psi$'s one is interested.

By a straightforward calculation one can verify that the final expressions for $\psi_1[J]$ and $\psi_2[J]$ in terms of propagators for the $\varphi^4$ theory are

$$\psi_1[J] = -\frac{1}{4!}[\langle \Delta_{xa}\Delta_{xb}\Delta_{xc}\Delta_{xd}J_aJ_bJ_cJ_d\rangle - 6\langle \Delta_{xx}\Delta_{xa}\Delta_{xb}J_aJ_b\rangle + 3\langle \Delta_{xx}^2\rangle], \tag{50}$$

$$\begin{aligned}\psi_2[J] = &-\frac{1}{2}\left\langle J_a\Delta_{ax}\left(\frac{1}{6}\Delta_{xy}^3 + \frac{1}{4}\Delta_{xx}\Delta_{xy}\Delta_{yy}\right)\Delta_{yb}J_b\right\rangle_{xyab}\\ &- \frac{1}{8}\langle J_a\Delta_{ax}\Delta_{xy}^2\Delta_{yy}\Delta_{xb}J_b\rangle_{xyab} + \frac{2}{4!}\langle J_a\Delta_{ax}\Delta_{xx}\Delta_{xy}\Delta_{yb}\Delta_{yc}\Delta_{yd}J_bJ_cJ_d\rangle_{xyabcd}\\ &+ \frac{3}{2(4!)}\langle J_aJ_b\Delta_{ax}\Delta_{bx}\Delta_{xy}^2\Delta_{yc}\Delta_{yd}J_cJ_d\rangle_{xyabcd}\\ &- \frac{1}{2(3!)^2}\langle J_aJ_bJ_c\Delta_{xa}\Delta_{xb}\Delta_{xc}\Delta_{xy}\Delta_{yd}\Delta_{ye}\Delta_{yf}J_dJ_eJ_f\rangle_{xyabcdef}\\ &+ \frac{1}{48}\langle \Delta_{xy}^4\rangle_{xy} + \frac{3}{2(4!)}\langle \Delta_{xx}\Delta_{xy}^2\Delta_{yy}\rangle_{xy}.\end{aligned} \tag{51}$$

The expressions for higher order $\psi$'s are increasingly more lengthy, but amenable to a systematic derivation by the above procedure. This we have done by developing a REDUCE program which confirms the results given above.

*2.1.2. The Hopf algebra $K\langle\{S_k\}\rangle$*

As we have seen in Eq. (12) the power sum symmetric functionals $\psi_k[J]$ appear naturally in PQFT as a result of a perturbative expansion (in the coupling parameter of the theory) of the transition amplitude. In terms of the $S_i[J]$ these $\psi_k[J]$ are given (cf. Eq. (13)) by the Schur polynomials. Using as generators the $S_i$'s, one can construct a universal enveloping algebra $K\langle\{S_k\}\rangle$ by introducing a Poincaré–Birkhoff–Witt basis $\{1, S_{i_1}, S_{i_1}S_{i_2}, \dots\}$, and defining multiplication $m$ as the disjoint union of the elements of this basis. Further, a coalgebra structure can be generated by defining the coproduct

$$\Delta(S_k) = \sum_{i=0}^{k} S_i \otimes S_{k-i}, \quad S_0 \equiv \mathbf{1} \tag{52}$$

and a counit $\varepsilon$ as the augmentation of the algebra by

$$\varepsilon(S_0) = 1, \qquad \varepsilon(S_k) = 0, \quad k \neq 0. \tag{53}$$

We can give this coalgebra the structure of a Hopf algebra by additionally defining an antipode $S$ as the involutive homomorphism (because of commutativity)

$$S(S_k) = -S_k - m(S \otimes \mathrm{id})\tilde{\Delta}(S_k), \tag{54}$$

where $\tilde{\Delta}$ is the coproduct operation with the primitive contributions removed.

Since $K\langle\{S_k\}\rangle$ is commutative, by the Milnor–Moore theorem there is a cocommutative Hopf algebra in duality with it which is necessarily isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{L})$, where $\mathfrak{L}$ is a Lie algebra. The generators $Z_i$ of $\mathfrak{L}$ are infinitesimal characters of $K\langle\{S_k\}\rangle$, i.e. they are linear mappings $Z_i : K\langle\{S_k\}\rangle \to K\langle\{S_k\}\rangle$ fulfilling the conditions:

$$\langle Z_i, S_k\rangle = \delta_{ik}, \tag{55}$$

$$\langle Z_i, S_k S_l\rangle = \langle Z_i, S_k\rangle\varepsilon(S_l) + \varepsilon(S_k)\langle Z_i, S_l\rangle. \tag{56}$$

We shall denote [15] by $\partial\,\mathrm{Char}\,K\langle\{S_k\}\rangle$ the set of infinitesimal characters of $K\langle\{S_k\}\rangle$. Note that $\mathfrak{L}$ is Abelian, since the coproduct in $K\langle\{S_k\}\rangle$ is cocommutative.

The exponential mapping $\sum_i \alpha_i Z_i \to \exp(\sum_i \alpha_i Z_i) \in \mathcal{G}$, equipped with a convolution product $*$ and unit $\mathbf{1}_*$:

$$\langle \chi * \eta, S_k\rangle = \langle \chi \otimes \eta, \Delta S_k\rangle, \quad \chi, \eta \in \mathcal{G}, \tag{57}$$

$$\langle \mathbf{1}_*, S_k\rangle = \varepsilon(S_k), \tag{58}$$

together with the inverse

$$\chi^{-1} = \chi \circ S, \tag{59}$$

generates a subgroup $\mathcal{G}$ of the group of characters of $K\langle\{S_k\}\rangle$.

$\mathcal{G}$ is dual to the Hopf algebra $K\langle\{S_k\}\rangle$ and it is multiplicative, i.e. it satisfies

$$\langle \chi, S_k S_l\rangle = \langle \chi, S_k\rangle\langle \chi, S_l\rangle, \quad \chi \in \mathcal{G}. \tag{60}$$

Moreover, from (57) and the fact that the Lie algebra of $\partial\,\mathrm{Char}\,K\langle\{S_k\}\rangle$ is Abelian, we have that the convolutive product in our case reduces to

$$\mathrm{e}^{\sum_i \alpha_i Z_i} * \mathrm{e}^{\sum_j \beta_j Z_j} = \mathrm{e}^{\sum_i (\alpha_i + \beta_i) Z_i}. \tag{61}$$

### 2.1.3. The Hopf algebra $K\langle\{\psi_k\} >$

The Hopf algebra $K\langle\{S_k\}\rangle$ induces another Hopf algebra $K\langle\{\psi_k\}\rangle$ by applying a change to normal coordinates [10] to our original Poincaré–Birkhoff–Witt basis, constructed from the $S_i$ coordinates, by means of the map with the canonical element $\mathbf{C} = \mathrm{e}^{\sum_i Z_i \otimes \psi_i}$ which acts as an identity map, i.e.:

$$\langle \mathrm{e}^{\sum_i Z_i \otimes \psi_i}, S_k \otimes \mathrm{id}\rangle = S_k. \tag{62}$$

It is not difficult to verify that the nonlinear relation between the $\psi_k$'s and $S_i$'s resulting from (62) is given by (13), and that the $\psi_k$'s acquire a primitive coproduct

$$\Delta\psi_k = \mathbf{1} \otimes \psi_k + \psi_k \otimes \mathbf{1} \tag{63}$$

and an antipode given by

$$S(\psi_k) = -\psi_k. \tag{64}$$

Note that since (13) is invertible $K\langle\{S_k\}\rangle \simeq K\langle\{\psi_k\}\rangle$.

As a parenthetical remark we point out that the normal coordinates that we construct here differ from those recently discussed in [10] in relation to the Hopf algebra of rooted trees, by the fact that in the context of the latter the comultiplication given by (63) corresponded to nonbranched trees, and that for the more general case of branched trees it was determined by application of the Baker-Hausdorff–Campbell formula.

## 2.2. Hopf primitives and connected Green functions

In order to establish explicitly the relation of the $\psi_k[J]$ functionals to the Green functions, observe that because the $n$-legged connected Green functions in PQFT are obtained from the functional variation

$$G_{\mathrm{E}}^{(n)}(x_1, \dots, x_n) = -\left.\frac{\delta^n Z_{\mathrm{E}}[J]}{\delta J_1 \cdots \delta J_n}\right|_{j=0}, \tag{65}$$

we have that

$$\psi_k[J] = \frac{1}{\lambda^k}\sum_{n=0}\frac{1}{(n)!}\langle G_k^{(n)}(x_1, \dots, x_n) J_1 \dots J_n\rangle, \tag{66}$$

where $G_k^{(n)}$ are the Euclidean Green functions resulting from adding all the connected Feynman graphs with $k$ vertices and $n$ external legs ($n$ is even for the $\varphi^4$ theory). Note also that (66) contains contributions from the Green functions $G_k^{(0)}$ which correspond to vacuum terms. These contributions may be absorbed in the $\ln W_0[0]$ term in (12) via the normalization constant $N$.

The analytical expressions for the graphs composing the Green functions result, in general, in ultraviolet divergences, and the standard procedure for removing them is to first apply dimensional regularization and then successively the Forest Formula. It is at this stage where the Kreimer–Connes Hopf algebra formalism provides an important insight into the underlying mathematics behind the Forest Formula of renormalization. Indeed, by noting that each Feynman diagram corresponds to a decorated rooted tree (or a sum of decorated rooted trees in the case of overlapping diagrams) and that these rooted trees, as well as the diagrams themselves, are the generators of respective universal enveloping Hopf algebras, these authors have shown that the operation with a twisted antipode on the algebra provides a systematic application of the Forest Formula and, consequently, a systematic procedure for generating the counterterms needed by the theory in order to cancel the unwanted infinities. Connes and Kreimer [5] further show the relation between the algebra of characters, dual to their Hopf algebra, and the Birkhoff algebraic decomposition. A detailed exposition of these ideas may be found in the papers cited in Section 1.

Here we only stress once more the fact that in our approach we deal with the Hopf primitives $\psi_k \in \mathfrak{L}_K(Y)$ rather than with Feynman graphs directly. As we have seen, these Hopf primitives are Lie polynomials where each monomial is a Hall tree that itself corresponds to a sum of Feynman diagrams with $k$-vertices each and an equal number of external legs and loops. This last observation can be read off directly from the word composed by the foliage of a Hall tree. First because the number of external legs $E$ in the Feynman diagrams constituting a Hall tree is given by

$$E = 2N_{Z^0} - N_{\delta/\delta J}V, \tag{67}$$

where $N_{Z^0}$ is the number of times the letter $Z^0$ appears in the word, $N_{\delta/\delta J}$ the degree of the functional derivation in the potential, $V$ the number of vertices which is equal to number of times the letter $\langle V\rangle$ appears in the word, so all Feynman diagrams composing a given Hall tree have the same number of external legs. Second, because the topology of the Feynman diagrams implies that the number of loops is fixed by the number of vertices and the number of external legs by the relation

$$(N-2)V = E + 2L - 2, \tag{68}$$

where $N$ is the total number of legs per vertex ($N = 4$ for the $\varphi^4$ theory) and $L$ the number of loops, so also all Feynman diagrams composing a Hall tree have the same loop number.

## 3. Discussion

As already mentioned in Section 1, given that a theory is renormalizable, the Hopf Algebra of Renormalization developed by Connes and Kreimer provides insight into the mathematical structures behind the Forest Formula and is extremely useful in numerical calculations since it gives a systematic procedure (amenable to computer programming [16]) for evaluating the renormalized Green functions. In a parallel vein, the Hopf algebra $K\langle Y\rangle$ investigated here, helps to exhibit some complementary mathematical structures associated with the Hausdorff series expansion of perturbation theory. We have shown the relation of the $\psi_k[J]$'s to the Hall polynomials, and that of their graphic representation in terms of Hall trees to the Feynman diagrams. Also, making use of the cyclic vector character of the $\psi_k[J]$'s with respect to the derivation endomorphism $D$, which appears as part of the structure of the Lie subalgebra of $K\langle Y\rangle$, we have obtained a systematic procedure (also subject to computer programming) for constructing all the Green functions of the theory starting from the generator $Z^0[J]$. Moreover, since each action of $D$ introduces an additional vertex in the commutators conforming the $\psi_k[J]$'s, one could hope that a further study of this operator and its possible deformations would result in further mathematical insights on the process of renormalization.

Note, however, that because of the Hopf primitive character of our $\psi_k[J]$'s and their relation to the Green functions given by (66) one should not expect that the application of the algebraic Birkhoff decomposition to our Hopf algebra should be immediately related to the Forest Formula. The Hopf Algebra of Renormalization begins where our Hopf algebra ends. Nonetheless, a unital $K$-algebra homomorphism $\phi : \{K\langle Y\rangle\} \to \mathbf{A}$ from the free algebra $K\langle Y\rangle$ to the (unital) $K$-algebra $\mathbf{A}$ of meromorphic functions on the Riemann sphere with poles at the origin, followed by a the Birkhoff decomposition allows us to exhibit more explicitly and in a heuristic fashion the point made in the second paragraph of Section 1. Indeed [15], if we let $\mathbf{A} = \mathbf{A}_- \oplus \mathbf{A}_+$ be a Birkhoff sum of the $K$-linear multiplicative subspaces $\mathbf{A}_-$ and $\mathbf{A}_+$, where $\mathbf{A}_- = \{\text{polynomials in } z^{-1} \text{ without constant term}\}$ and $\mathbf{A}_+ = \{\text{restriction to } (\mathbb{C} - \{0\}) \text{ of functions in } \mathbf{Holom}(\mathbb{C})\}$, and if we further let $T : \mathbf{A} \to \mathbf{A}_-$ be the Rota-Baxter projection operator, satisfying the multiplicative constraints

$$T(ab) + (Ta)(Tb) = T[(Ta)b + a(Tb)], \quad a, b \in \mathbf{A}, \tag{69}$$

then the algebraic Birkhoff decomposition

$$\phi_+ = \phi_- \star \phi, \tag{70}$$

(where the operator $\star$ denotes the convolution product $(\phi\star\phi')(w) = m_{\mathbf{A}}(\phi \otimes \phi')(\Delta w)$, $\phi, \phi' \in \mathrm{Hom}_{K-\mathrm{alg}}(K\langle Y\rangle, \mathbf{A})$), together with the Hopf primitive character of the $\psi_k[J]$'s immediately imply that

$$\phi(\psi_k[J]) = \phi_+(\psi_k[J]) - \phi_-(\psi_k[J]) = \phi_+(\psi_k[J]) + T\phi(\psi_k[J]). \tag{71}$$

Thus, by virtue of (66),

$$\begin{aligned}\phi_+(\psi_k) = \frac{1}{\lambda^k} \sum_{n\geq 1} \frac{1}{(n)!} (\langle G_k^{(n)}(x_1, \dots, x_n) J_1 \dots J_n\rangle \\ - \langle T[G_k^{(n)}(x_1, \dots, x_n)] J_1 \dots J_n\rangle),\end{aligned} \tag{72}$$

where we have put the projector $T$ inside of the integration in the second term of (72) after taking into account that the currents $J_i$ are good test functions, so the pole structure of the integrals is determined by that of the Green functions. In addition, the projection by $T$ to $\mathbf{A}_-$ of the connected Green functions is taken in the Mass Independent Renormalization Scheme.

Recall now the basic equation of the renormalization group which relates the dimensionally regularized bare and renormalized connected Green functions, and which in our notation reads:

$$\frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_n} \sum_{k\geq 0} (\lambda_b)^k (\psi_k[J])_b = \frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_n} \sum_{k\geq 0} \lambda^k (\psi_k[J])_R. \tag{73}$$

In the above, the subscript $b$ denotes that the regularized Green functions in the momentum representation, occurring in the $\psi_k[J]$'s, are expressed in terms of the bare parameters of the theory, i.e. $(\psi_k[J])_b = \sum_{n=0} (\psi_k^{(n)}[J])_b = (1/\lambda_b^k) \sum_{n=0} (1/(n)!) \langle (G_k^{(n)})_b(p_1, \dots, p_n;$ $\lambda_b, m_b, \epsilon)$, $\lambda_b = Z_\lambda \lambda$, $m_b^2 = Z_m m^2$, $(J_k)_b = Z_\varphi^{-1/2} J_k$, while the subscript $R$ on the right side of (73) denotes the $\psi_k$'s evaluated with the renormalized Green functions. Furthermore, in the Mass Independent Scheme the counterterms contain no finite contributions and are polynomials in inverse powers of $\epsilon$ with coefficients depending only on $\lambda$, so the renormalization parameters $Z_\lambda$, $Z_\varphi$ and $Z_m$ are polynomials in inverse powers of $\epsilon$ with coefficients depending only on $\lambda$.

If we replace (71) in (73) and observe that the right side of the later equation is finite, we thus arrive at the following set of equations:

$$\lim_{\epsilon\to 0}\left[\text{Finite}\left\{\sum_{k\geq 0}(Z_\lambda)^k Z_\varphi^{-n/2}(G_k^{(n)}(p_1,\ldots,p_n;\lambda,Z_m m^2,\epsilon))_{(+)}\right\}\right] = \sum_{k\geq 0}(G_k^{(n)}(p_1,\ldots,p_n;\lambda,m,)_R \tag{74}$$

and

$$\text{Poles}\left\{\sum_{k\geq 0}(Z_\lambda)^k Z_\varphi^{-n/2}(G_k^{(n)}(p_1,\ldots,p_n;\lambda,Z_m m^2,\epsilon))_{(-)}\right\} = 0, \tag{75}$$

where

$$(G_k^{(n)}(p_1,\ldots,p_n;\lambda,Z_m m^2,\epsilon))_{(+)} := [G_k^{(n)}(p_1,\ldots,p_n,Z_m m^2,\epsilon) - T(G_k^{(n)}(p_1,\ldots,p_n,Z_m m^2,\epsilon)] \tag{76}$$

and

$$(G_k^{(n)}(p_1,\ldots,p_n;\lambda,Z_m m^2,\epsilon))_{(-)} := T(G_k^{(n)}(p_1,\ldots,p_n,Z_m m^2,\epsilon). \tag{77}$$

But if, and only if, the theory is renormalizable, Eq. (75) can be solved consistently and order by order in the powers of $\lambda$, i.e. only then the counterterms can be absorbed into the parameters of the theory, and only then the theory will be physical. Formally, the substitution of the resulting renormalization functions into the left side of (74) would then yield the physically meaningful renormalized Green functions. In such a case, however, the standard recursive elimination of subdivergences via the Forest Formula (or the Connes–Kreimer Hopf algebra) would clearly be a more efficient manner to derive these quantities. On the other hand, the opposite is not necessarily true: The fact that the forest formula provides a finite answer does not suffice to make the theory physical, it is also necessary that the poles of the ill-defined regularized Green functions can be absorbed into the bare parameters of the theory.

To conclude, we would like to comment on several other possible lines for future work in addition to the ones already mentioned. Thus, for example, since there is a duality between concatenation and shuffle products [14,18], there are actually two bialgebra structures on $K\langle Y\rangle$: The one considered here with product (words) made out by concatenation of letters of the alphabet $Y$ and coproduct defined by (16–18), the other one with shuffle product and coproduct defined by

$$\Delta'(w) = \sum_{u,v\in Y^*}(w,uv)u\otimes v, \tag{78}$$

where $Y^*$ is the free monoid on $Y$. It may be interesting to investigate the relation of this other bialgebra to the primitives $\psi_k[J]$ discussed here, and if this relation has a possible relevance to renormalization.

One could also ask if it is possible to carry out a program starting from the Hopf algebra of Feynman diagrams and perform a Legendre transformation to a Hopf algebra of $\psi_k$, which would no longer be primitive, and by exponentiation to a Hopf algebra of $S_k$. It would be interesting then to consider not only the Hopf algebra $K\{\psi_k\}$ but also $K\{\psi_k\}/\sim$, where $\sim$ stands for an equivalence relation coming, e.g. from a Ward identity. Some of these lines of work are under present consideration and the results will be reported elsewhere.

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